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SOLUTION BY ISAAC H. TURRELL, CINCINNATI, OHIO.

The standard case will be when the contacts are all external; for instance when three circles, touching one another externally, are inscribed within the space enclosed by three others, also tangent to one another externally, each circle of either group touching two of the other group, and conversely.

Denoting the circles of the outer group and their centers by a, b, c , and of the inner by x, y, z , there will be three pairs of non-touching circles, viz; ax, by, cz , and it is to be shown that the lines ax, by, cz are concurrent.

Let P, P', P'', O, O', O'' , be the external centers of similitude of the pairs ab, ac, bc, ax, by, cz , respectively; and since, when two circles touch in the same way, another pair of circles, the external center of similitude of either pair lies on the radical axis of the other pair, the points O'', O', P'' , lie on the radical axis of ax , and O'', O, P' , lie on the radical axis of by ; similarly the points O', O, P , lie on the radical axis of cz .

Again, the radical axis of two circles is perpendicular to the line joining their centers, and these radical axes form the triangle $OO'O''$; hence (since the perpendiculars of a triangle meet in a point), the lines joining the centers of the non-touching circles are concurrent.

In case of *internal* contact among any of the circles, the same method of proof will apply, if the corresponding *internal* centers of similitude are employed instead of the external centers.

NOTE.—The foregoing properties of circles in contact combined with the "Theorem of Pascal applied to the Circle," affords a simple and remarkably elegant solution, by pure geometry, of the following noted problem, namely:

Within the space enclosed by three circles which touch one another externally, to inscribe three circles tangent to one another, each of which shall touch two of the given circles.

[Miss Ladd has submitted an elaborate and very elegant solution of this question which we hope to publish in a future number.]

188. "Two given rectangular planes $ABFE$ and $EFCD$ lie contiguous, as in the figure. The plane BE is hard ground whereon one can travel at the rate $2m$; the other is soft ground whereon the rate of travel is m .

Required the point O of crossing EF so that the time of journey from A to C shall be the quickest possible."

SOLUTION BY THE EDITOR.

Take a point O' on the line EF , join AO' and $O'C$ and draw $O'g$ and Oh respectively perpendicular to AO and CO' , then is $\angle OO'g = \angle OAE$.

Suppose OO' to be diminished to infinity, then will the ultimate ratio of Ag to AO' , of Ch to CO , and of $\angle O'Oh$ to $\angle OCF$ be a ratio of equality; and therefore the time in Ag will equal the time in AO' , and the time in Ch will equal the time in CO , therefore the time in Og must equal the time in $O'h$; but the rate in Og is $2m$ while the rate in $O'h$ is m , $\therefore$

$$Og = 2O'h \quad (1)$$

And because of the equality of the angles OAE and $OO'g$, and OCF and $O'Oh$, if we put $OO' = r$, we shall have $Og = r \sin OAE$ and $O'h = r \sin OCF$; $\therefore$, by (1),

$$\sin OAE = 2 \sin OCF. \quad (2)$$

Put $AO = R$ and $CO = R'$, then we have $OE = R \sin OAE$ and $OF = R' \sin OCF = R' \frac{1}{2} \sin OAE$ by (2), $= \frac{1}{2} R' \sin OAE$;

$$\therefore R \times OF = \frac{1}{2} R' \times OE. \quad (3)$$

If we put $AE = a$, $ED = b$, $EF = c$ and $EO = x$, we readily get $R = \sqrt{(a^2 + x^2)}$, and $R' = \sqrt{[b^2 + (c - x)^2]}$, which values substi'd in (3) give

$$\sqrt{(a^2 + x^2)}(c - x) = \frac{1}{2} \sqrt{[b^2 + (c - x)^2]} x,$$

an equation of the fourth degree from which the value of x may be found.

SOLUTION BY GEO. H. BANGS, NEW YORK CITY.

Let the route AOC be that of the least time. Produce AO to H and CO to G . (G being a point in DA produced.) Then if the rate on OH = the rate on OA the line AH will give the least time from A to H ; and if the rate on OG = the rate on OC the line CG will give the least time from C to G . Now it is plain that the lines AH and CG may be taken as the representatives of *times* for units of space. Therefore, in this case, $AH : CG :: \frac{1}{2}m : m$; or $AH = \frac{1}{2}CG$, whence it readily appears that $OH = \frac{1}{2}OC$ and $OA = \frac{1}{2}OG$. Hence, putting $AE = a$, $DE = b$, $EF = c$ and $EO = x$, we get

$$\sqrt{(x^2 + a^2)}(x - c) = \frac{1}{2} \sqrt{[x^2 + (x - c)^2]} x.$$

[Mr Seitz also gave a solution of this question without formal application of Calculus.]

189. "Find by Algebra the series whose sum is 127, when the sum of the squares of the seven terms is 5461."

SOLUTION BY PROF. J. H. KERSHNER, MERCERSBERG, PA.

$$\text{Let} \quad x + xy + xy^2 + xy^3 + xy^4 + xy^5 + xy^6 = 127, \quad (1)$$

$$x^2 + x^2y^2 + x^2y^4 + x^2y^6 + x^2y^8 + x^2y^{10} + x^2y^{12} = 5461. \quad (2)$$

Dividing (2) by (1) we get

$$x - xy + xy^2 - xy^3 + xy^4 - xy^5 + xy^6 = 43; \quad (3)$$

adding (1) and (3) and taking half their sum, we have

$$x + xy^2 + xy^4 + xy^6 = 85. \quad (4)$$

$$\text{Subt. (3) from (4), } xy + xy^3 + xy^5 = 42. \quad (5)$$

Multiply (5) by y and subtract the product from (4) and we get

$$x = 85 - 42y, \text{ or } y = 2 + [(1-x) \div 42]. \quad \therefore x = 1, \text{ and } y = 2.$$

190. "Transform the equation, $x^9 + 9x^8 + 36x^7 + 85x^6 + 132x^5 + 141x^4 + 105x^3 + 54x^2 + 18x + 9 = 0$, into an equation of the third degree."

SOLUTION BY DAVID S. HART, A. M., M. D., STONINGTON, CONN.

Subtract $(x+1)^9 = 0$ from the given equation, and there remains

$$x^6 + 6x^5 + 15x^4 + 21x^3 + 18x^2 + 9x + 8 = 0. \quad (2)$$

Subtract $(x+1)^6 = 0$ from (2) and there remains

$$x^3 + 3x^2 + 3x + 7 = 0; \quad (3)$$

and from (3) subtract $(x+1)^3 = 0$ and there remains $6 = 0$.

Therefore $(x+1)^9 + (x+1)^6 + (x+1)^3 + 6 = 0$, or, putting $(x+1)^3 = y$,

$$y^3 + y^2 + y + 6 = 0,$$

which is an equation of the third degree.

191. "Compute, in the most convenient way, the product of the differences of the equation $x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$."

SOLUTION BY A. S. HATHAWAY, ITHACA, NEW YORK.

The product of the differences of the roots is equal to the square root of the product of the squares of the differences, but this last product is the discriminant of the equation.

Let $x = y \div z$, then multiply by z^5 and we obtain

$$y^5 - ay^4z + by^3z^2 - cy^2z^3 + dyz^4 - ez^5 = 0. \quad (1)$$

Differentiate with respect to y , and also with respect to z ;

$$5y^4 - 4ay^3z + 3by^2z^2 - 2cyz^3 + dz^4 = 0, \quad (2)$$

$$-ay^4 + 2by^3z - 3cy^2z^2 + 4dyz^3 - 5ez^4 = 0. \quad (3)$$

Multiply (2) by $-a$, (3) by 5, subtract, and divide $dy \ z$;

$$(10b - 4a^2)y^3 + (3ab - 15c)y^2z + (20d - 2ac)yz^2 + (ad - 25e)z^3 = 0. \quad (4)$$

Again, multiply (2) by $-ay + 2bz$, (3) by $5y - 4az$, subt., and divide by z^2 ;

$$(3ab-15c)y^3+(20d+10ac-6b^2)y^2z+(4bc-15ad-25e)yz^2+(20ae-2bd)z^3=0. \quad (5)$$

Now multiply (2) by $-ay^2+2byz-3cz^2$, (3) by $5y^2-4ayz+3bz^2$, subtract, and divide by z^3 , and we obtain

$$(20d-2ac)y^3+(4bc-15ad-25e)y^2z+(20ae-10bd-6c^2)yz^2+(3cd-15be)z^3=0. \quad (6)$$

Lastly, multiply (2) by $-ay^3+2by^2z-3cyz^2+4dz^3$, (3) by $5y^3-4ay^2z+3byz^2-2cz^3$, subtract, and divide by z^4 , and we get

$$(ad-25e)y^3+(20ae-2bd)y^2z+(3cd-15be)yz^2+(10ce-4d^2)z^3=0. \quad (7)$$

Now the discriminant of (1) is the resultant of (2) and (3) which is the determinant of the coefficients of (4), (5), (6), (7); but this discriminant is equal to the product of the squares of the differences of the roots; therefore for the product of the differences of the roots we obtain the following expression:

$$\begin{vmatrix} 10b-4a^2 & 3ab-15c & 20d-2ac & ad-25e \\ 3ab-15c & 20d+10ac-6b^2 & 4bc-15ad-25e & 20ae-2bd \\ 20d-2ac & 4bc-15ad-25e & 20ae+10bd-6c^2 & 3cd-15be \\ ad-25e & 20ae-2bd & 3cd-15be & 10ce-4d^2 \end{vmatrix}^{\frac{1}{2}}$$

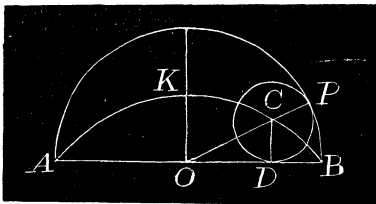
192 "Find the average area of all the circles inscribed in a given semicircle."

SOLUTION BY PROF. H. T. J. LUDWICK, MT. PLEASANT, N. C.

Let O represent the center of the given semicircle and C the center of an inscribed circle. Put $AO = a$, $OD = x$ and $DC = y$.

The locus of C is $x^2+2ay = a^2$. Area of circle $= \pi y^2 = \pi(a^2 - x^2)^2 \div 4a^2$.

Length of arc $AKB = \frac{1}{a} \int_0^a (a^2 + x^2)^{\frac{1}{2}} dx$. Denoting by A the average area,



$$\begin{aligned} A &= \frac{\pi}{4a^2} \int_0^a (a^2 - x^2)^2 (a^2 + x^2)^{\frac{1}{2}} dx \div \int_0^a (a^2 + x^2)^{\frac{1}{2}} dx \\ &= \frac{\pi a^2}{96} \left(\frac{39 \log(1+\sqrt{2}) - 5\sqrt{2}}{\sqrt{2} + \log(1+\sqrt{2})} \right). \end{aligned}$$

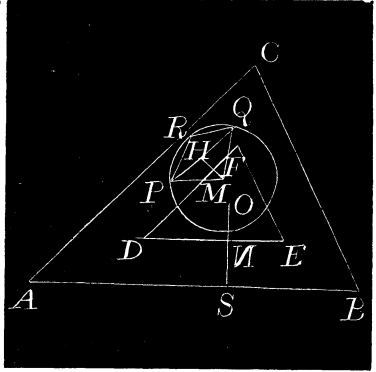
193. "A triangle is formed by joining three points taken at random in the surface of a given triangle. Find the chance that the circle circumscribing this triangle lies wholly within the given triangle."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let ABC be the given triangle, PQR the triangle formed, O the center of the inscribed circle of ABC , and M the center of the circle circ'bing PQR .

Draw the triangle DEF , making its sides parallel to those of ABC , and at a distance from them equal to MP , and draw ONS and MH perpendicular to AB and PQ .

Now while PQ is given in length and direction, and the angle PRQ is given, if MP is less than the radius of the inscribed circle of ABC , the area of the trian. DEF represents the number of ways the three points can be taken, so that the circumscribed circle will lie wholly within the given $\triangle$.



Let $PQ = 2x$, $OS = r$, perimeter of $ABC = s$, area of segment $PRQ = t$, area of sector $PMQ = v$, area of $\triangle PMQ = u$, area of $\triangle ABC = \Delta$, $\angle PMH = \theta$, $\varphi = \sin^{-1}(x \div r)$, and $\psi =$ the angle which PQ makes with some fixed line. Then we shall have $PM = x \operatorname{cosec} \theta$, $ON = r - x \operatorname{cosec} \theta$, area $DEF = (r - x \operatorname{cosec} \theta)^2 (\Delta \div r^2)$, $v = \theta x^2 \operatorname{cosec}^2 \theta$, $u = x^2 \cot \theta$, $t = v - u$. An element of the triangle at Q is $4x dx d\psi$, or $4r^2 \sin \varphi \cos \varphi d\varphi d\psi$, and at R it is dt . The limits of ψ are 0 and 2π ; of θ , φ and $\pi - \varphi$; and of x , 0 and r , or of φ , 0 and $\frac{1}{2}\pi$. Hence, doubling, since R may lie on either side of PQ , we have for the required chance

$$p = \frac{2}{\Delta^3} \int_0^r 4x dx \int_{\theta=\varphi}^{\theta=\pi-\varphi} dt \int_0^{2\pi} d\psi (r - x \operatorname{cosec} \theta)^2 \frac{\Delta}{r^2} = \frac{2\pi^2 r^4}{5\Delta^2} = \frac{8\pi^2 r^2}{5s^2}.$$

194. "Sum the series $1 - \frac{1}{2^2} + \frac{1}{4^2 \cdot 2^2} - \frac{1}{6^2 \cdot 4^2 \cdot 2^2} + \frac{1}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2} - \&c.$ "

SOLUTION BY PROF. J. SCHEFFER.

If by means of the formula

$$\int \sin^n x dx = \frac{-\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx}{n},$$

we develop the successive integrals $\int_0^\pi \sin^2 x dx$, $\int_0^\pi \sin^4 x dx$, $\int_0^\pi \sin^6 x dx$, etc., we find that the given series is equal to the definite integral

$$\frac{1}{\pi} \int_0^\pi \cos(\sin x) dx.$$

The series converges rapidly. Eight terms suffice to obtain the sum correctly for at least 10 decimals. The sum is $= .7651976865$.

Such integrals as the last are always, and without difficulty, obtained by series.

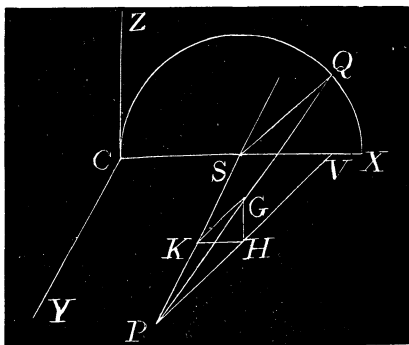
195. "A uniform rod rests with one extremity against a rough vertical wall, and with the other extremity on a rough horizontal plane, such that it is held in equilibrium by friction alone. The beam is *not* in a vertical plane. The coefficient of friction of the horizontal plane and vertical wall being respectively μ and μ' ; find the normal pressure of the rod upon the horizontal plane and vertical wall, and the exact position of the rod with reference to the two latter planes."

SOLUTION BY GEO. EASTWOOD SAXONVILLE, MASS.

Let the plane XCZ represent the vertical wall and the plane XCY the horizontal plane.

Let PQ be a position of the rod when it is bordering on motion. Draw PS perpendicular to CX , and join SQ .

If we suppose for a moment, to fix our thoughts, that P is a fixed point, the locus of the point Q will be a semi-circle CQX , having the point S for its centre. Let G be the centre of gravity of the rod, PHV the projection of PQ in the plane of XY ; H and V being the projections of the points G and Q .



Draw HK parallel to CX , which will evidently be at right angles to PS , and join KG . Let W be the weight of the rod, μ' the coefficient of friction between the rod and vertical wall, and R' their mutual pressure; then the force $\mu'R'$ will act in the tangent to the locus at the point Q , that is, at right angles to SQ and in the plane CQX .

Put $PG = GQ = \beta$, $\angle QPS = \alpha$, $\angle QSV = \theta$, R = reaction of horizontal plane, and μR = the force of friction at P .

Now the horizontal forces acting on the rod are, μR at P , and $\mu'R' \sin \theta$ at Q ; $\therefore$

$$\mu R = \mu'R' \sin \theta. \quad (1)$$

The vertical forces acting on the rod are R at P and W at G , and $\mu'R' \times \cos \theta$ at Q ; $\therefore$

$$R + \mu'R' \cos \theta = W. \quad (2)$$

Next, conceive a line to be drawn perpendicular to PS , and let moments be taken about these two lines. Then

$$W.HK = \mu'R'.SQ, \quad (3)$$

$$W.PK = R'.QV + \mu'R' \cos \theta. \quad (4)$$

Geometrical relations give, $HK = GK \cos \theta = \beta \sin \alpha \cos \theta$, $SQ = 2\beta \sin \alpha$, $PS \cos \theta = 2\beta \cos \alpha \cos \theta$, $PK = \beta \cos \alpha$, $QV = SQ \sin \theta = 2\beta \sin \alpha \sin \theta$.

Making these substitutions in (3) and (4), the latter equations become

$$W\beta \sin \alpha \cos \theta = 2\beta \mu'R' \sin \alpha, \quad (5)$$

$$W\beta \cos \alpha = 2\beta R' \sin \alpha \sin \theta + 2\beta \mu'R' \cos \alpha \cos \theta. \quad (6)$$

Dividing (6) by (5), and reducing, we finally obtain $\tan \alpha = \mu' \tan \theta$.

From eq's (1) and (2), $\sin \theta = \mu R \div \mu' R' \dots (7)$, $\cos \theta = (W - R) \div \mu' R'$. (8)
Hence $\tan \alpha = \mu' \mu R \div (W - R)$, and therefore

$$\sin \alpha = \frac{\mu' \mu R}{[(W - R)^2 + (\mu' \mu R)^2]^{\frac{1}{2}}} \dots (9), \cos \alpha = \frac{W - R}{[(W - R)^2 + (\mu' \mu R)^2]^{\frac{1}{2}}}. \quad (10)$$

The substitution of (8) in (5) gives

$$W \frac{W - R}{\mu' R'} = 2\mu' R', \dots W - R = \frac{2(\mu' R')^2}{W} \dots (11), \text{ and } R = \frac{W^2 - 2(\mu' R')^2}{W}$$

In (6), substitute the values of $\sin \theta$, $\cos \theta$, $\sin \alpha$, $\cos \alpha$, as found in (7), (8), (9) and (10) and reduce; then

$$W(W - R) = 2\mu^2 R^2 + 2(W - R)^2. \quad (13)$$

By means of (11) and (12), equation (13) becomes

$$2(\mu' R')^3 = 2\mu^2 \left(\frac{W^2 - 2(\mu' R')^2}{W} \right)^2 + \frac{8(\mu' R')^4}{W^2}, \text{ which reduces to}$$

$$R'^4 - \frac{(4\mu^2 + 1)W^2}{4\mu'^2(\mu^2 + 1)} R'^2 = - \frac{\mu^2 W^4}{4\mu'^4(\mu^2 + 1)}. \quad (14)$$

The resolution of equation (14) gives

$$R' = \pm \frac{W}{2\mu'} \left[\frac{(4\mu + 1) \mp (1 - 8\mu^2)^{\frac{1}{2}}}{2(\mu^2 + 1)} \right]^{\frac{1}{2}}.$$

This value of the reaction of the vertical plane, substituted in (12) determines the reaction of the horizontal plane, and by means of these values and the given coeff's μ' and μ , the angles α and θ can be readily found.

PROBLEMS.

196. BY P. M. MC. KAY, LOOXAHOMA, MISS.—The sides of a triangle are respectively $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, x being any number greater than one; prove that the angle opposite the side $x^2 + x + 1$ is equal to 120° .

197. BY LIEUT. FRED. SCHWATKA, SPOTTED TAIL AG'Y, D. T.— AB is the diameter of a circle whose center is C , D is the middle point of CB , and is the center of a circle whose diameter is CB , and E is the middle p't of AC . EF is perp. to AB and intersects the circle C in F , and EG is tangent to the circle D at G . Prove that EFG is an equilateral triangle.

198. BY ARTEMAS MARTIN, M. A., ERIE, PA.—Find the average dist. between two points taken at random in the surface of a rectan., sides a & b .

199. BY AMATEUR.—Find an infinite series the sum of which is a square,—the sum of n terms of which is a square,—and the sum of the remaining terms, after the n terms are taken away, is a square.